# Supplemental Material for "Deterministic All-Optical Quantum State Sharing"

Yingxuan Chen[1], Qiqi Zhu[1], Xutong Wang[1], Yanbo Lou[1*], Shengshuai Liu[1*], and Jietai Jing[1,2,3*]
[1] *East China Normal University, State Key Laboratory of Precision Spectroscopy, Joint Institute of Advanced Science and Technology, School of Physics and Electronic Science, Shanghai, China*
[2] *CAS Center for Excellence in Ultra-intense Laser Science, Shanghai, China*
[3] *Shanxi University, Collaborative Innovation Center of Extreme Optics, Taiyuan, China*

## 1. Additional Gaussian noise of (2, 3) threshold deterministic all-optical quantum state sharing (AOQSS) protocol.

Figure S1(a) (S1(b)) shows the noise power spectra for the amplitude (phase) quadrature variances of $\hat{a}_{\text{in}}$ (blue trace), $\hat{a}_1$ (orange trace), $\hat{a}_2$ (green trace), and $\hat{a}_3$ (grey trace). The amplitude (phase) quadrature variance of $\hat{a}_3$ is $5.04 \pm 0.08$ dB ($4.91 \pm 0.08$ dB) above that of the input secret state $\hat{a}_{\text{in}}$. The gain of parametric amplifier for generating Einstein-Podolsky-Rosen (EPR) entangled source is about 1.26. It means that the amplitude (phase) quadrature variances of $\hat{a}_3$ should be about 1.82 dB above that of the input secret state $\hat{a}_{\text{in}}$ in theory. In this sense, based on Eq. (3) of the main manuscript, we can calculate that the additional Gaussian noise of quantum state sharing is about 3.22 dB. The amplitude and phase quadrature variances of $\hat{a}_1$ ($\hat{a}_2$) are about $2.06 \pm 0.08$ dB and $1.98 \pm 0.08$ dB ($2.08 \pm 0.08$ dB and $2.14 \pm 0.08$ dB) above the input secret state $\hat{a}_{\text{in}}$, respectively.

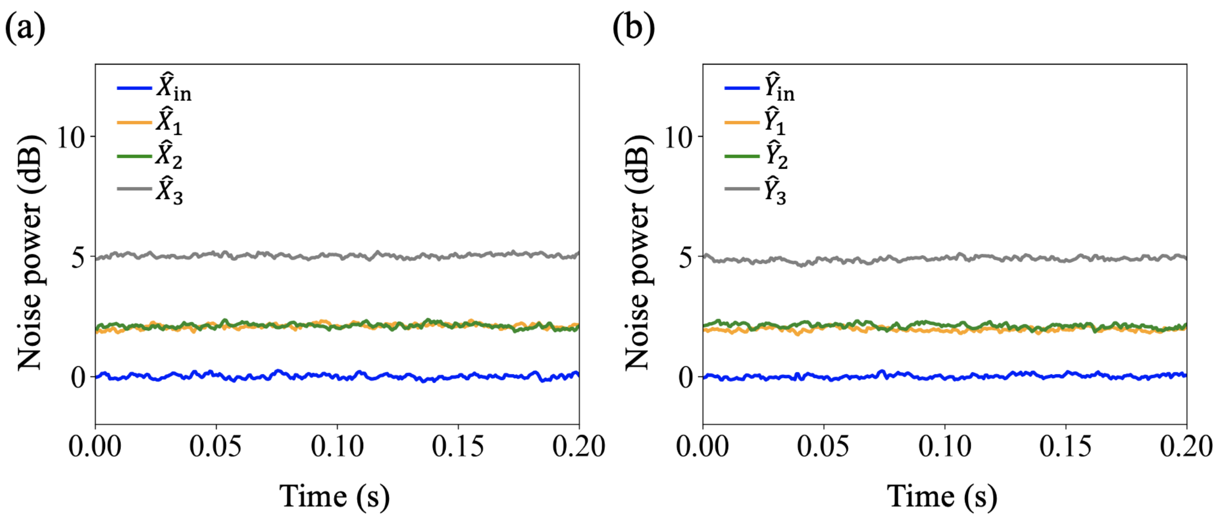


Figure S1: The noise power spectra for the (a) amplitude and (b) phase quadrature variances of $\hat{a}_{\text{in}}$ (blue traces), $\hat{a}_1$ (orange traces), $\hat{a}_2$ (green traces), and $\hat{a}_3$ (grey traces). The vertical scale is normalized to the quadrature variances of the input secret state.

*Electronic address: jtjing@phy.ecnu.edu.cn
ssliu@lps.ecnu.edu.cn
yblou@lps.ecnu.edu.cn

## 2. The detailed derivation of the fidelity of (2, 3) threshold deterministic AOQSS protocol.

Figure S2 shows the schematic of (2, 3) threshold deterministic AOQSS protocol. In order to implement the (2, 3) threshold deterministic AOQSS protocol, the dealer needs to encode the secret coherent state $\hat{a}_{\text{in}}$ into three shares with the help of EPR entangled source. This EPR entangled source is generated by four-wave mixing (FWM) process with the Hamiltonian given by [44]

$$\hat{H} = i\hbar r \hat{a}^{\dagger}_{\text{EPR1}} \hat{a}^{\dagger}_{\text{EPR2}} + \text{H.c.}. \tag{S1}$$

$\hat{a}^{\dagger}_{\text{EPR1}}$ and $\hat{a}^{\dagger}_{\text{EPR2}}$ are the creation operators associated with EPR1 and EPR2, respectively. $r$ denotes the interaction strength of FWM process. H.c. is the Hermitian conjugate. The output fields of the FWM process can be expressed as [44]

$$\begin{aligned} \hat{a}_{\text{EPR1}}(\tau) &= \sqrt{G_1}\hat{a}_{01} + \sqrt{G_1 - 1}\hat{a}^{\dagger}_{02}, \\ \hat{a}^{\dagger}_{\text{EPR2}}(\tau) &= \sqrt{G_1 - 1}\hat{a}_{01} + \sqrt{G_1}\hat{a}^{\dagger}_{02}, \end{aligned} \tag{S2}$$

where $\hat{a}_{01}$, $\hat{a}_{02}$ are the annihilation operators associated with the vacuum input states. $G_1 = \cosh^2(r\tau)$ is the intensity gain of the FWM process and $\tau$ is the interaction time. Then, $\hat{a}_{\text{in}}$ and $\hat{a}_{\text{EPR1}}$ are combined by a 50:50 beam splitter ($\text{BS}_1$). The outputs of $\text{BS}_1$ are sent to player1 and player2, and $\hat{a}_{\text{EPR2}}$ is sent to player3. The three shares with additional Gaussian noise can be expressed as [38]

$$\begin{aligned} \hat{a}_1 &= (\hat{a}_{\text{in}} - \hat{a}_{\text{EPR1}} - \delta N)/\sqrt{2}, \\ \hat{a}_2 &= (\hat{a}_{\text{in}} + \hat{a}_{\text{EPR1}} + \delta N)/\sqrt{2}, \\ \hat{a}_3 &= \hat{a}_{\text{EPR2}} + \delta N^*. \end{aligned} \tag{S3}$$

The additional Gaussian noise is denoted by $\delta N = (\hat{X}_N + i\hat{Y}_N)/2$, which has a mean of $\langle \hat{X}_N \rangle = \langle \hat{Y}_N \rangle = 0$ and variance of $\Delta^2 \hat{X}_N = \Delta^2 \hat{Y}_N = V_N$.

For (2, 3) threshold deterministic AOQSS protocol, at least two of these three players need to cooperate to reconstruct the secret state, and there are three reconstruction structures for (2, 3) threshold deterministic AOQSS. When player1 and player2 are authorized, the secret state can be reconstructed by $\{1, 2\}$ reconstruction structure, which is shown in Fig. S2(b). In this structure, $\hat{a}_1$ is combined with $\hat{a}_2$ by the 50:50 $\text{BS}_2$. By locking the relative phase between $\hat{a}_1$ and $\hat{a}_2$ to 0, the output beam can be expressed as [37]

$$\hat{a}_{\text{out}} = (\hat{a}_1 + \hat{a}_2)/\sqrt{2} = \hat{a}_{\text{in}}. \tag{S4}$$

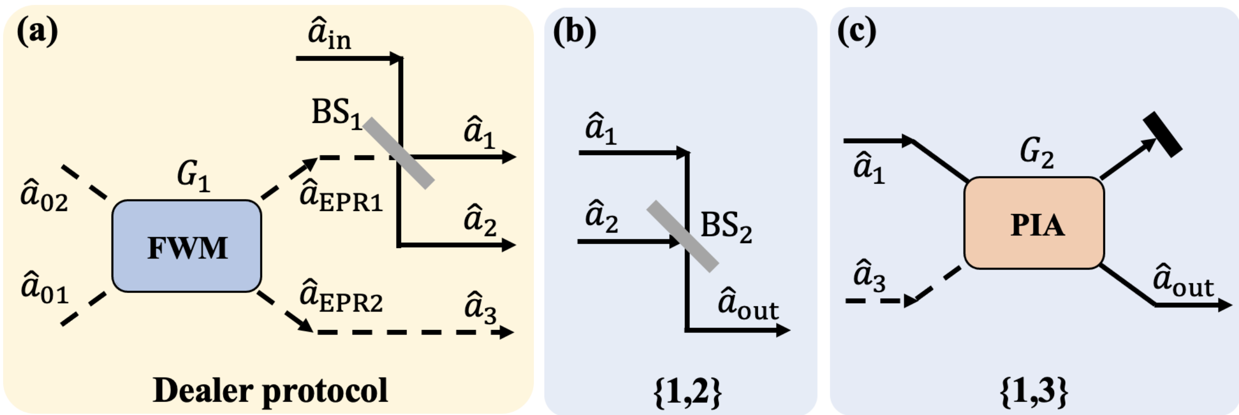


Figure S2: The schematic of (2, 3) threshold deterministic AOQSS protocol. (a) The dealer protocol. FWM, four-wave mixing; $G_1$, the intensity gain of the FWM process; $\hat{a}_{01}$ and $\hat{a}_{02}$, the annihilation operators associated with the vacuum input states of the FWM; $\hat{a}_{\text{EPR1}}$ and $\hat{a}_{\text{EPR2}}$, the annihilation operators associated with EPR1 and EPR2, respectively; $\hat{a}_{\text{in}}$, the annihilation operator associated with the secret coherent state; $\text{BS}_1$, 50:50 beam splitter; $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$, the annihilation operators associated with the three shares. (b) $\{1, 2\}$ reconstruction structure. $\text{BS}_2$, 50:50 beam splitter; $\hat{a}_{\text{out}}$, the annihilation operator associated with the output reconstructed state. (c) $\{1, 3\}$ reconstruction structure. PIA, phase-insensitive amplifier; $G_2$, the intensity gain of PIA.

The amplitude quadrature and phase quadrature of the output state are given by

$$\begin{aligned}\hat{X}_{\rm out} &= \hat{a}^{\dagger}_{\rm out} + \hat{a}_{\rm out} = \hat{X}_{\rm in},\\ \hat{Y}_{\rm out} &= i(\hat{a}^{\dagger}_{\rm out} - \hat{a}_{\rm out}) = \hat{Y}_{\rm in}.\end{aligned} \tag{S5}$$

Based on Eq. (S3) and Eq. (S4), we can see that the secret coherent state is retrieved by player1 and player2, while player3 can't get any information of the secret state. The quality of the reconstructed state is quantified by its fidelity [47], which can be expressed as [37, 38]

$$F = 2e^{-(k_x+k_y)/4}/\sqrt{(1+\Delta^2\hat{X}_{\rm out})(1+\Delta^2\hat{Y}_{\rm out})}, \tag{S6}$$

where $k_x = \langle\hat{X}_{\rm in}\rangle^2(1-g_x)^2/(1+\Delta^2\hat{X}_{\rm out})$, $k_y = \langle\hat{Y}_{\rm in}\rangle^2(1-g_y)^2/(1+\Delta^2\hat{Y}_{\rm out})$, $g_x = \langle\hat{X}_{\rm out}\rangle/\langle\hat{X}_{\rm in}\rangle$, and $g_y = \langle\hat{Y}_{\rm out}\rangle/\langle\hat{Y}_{\rm in}\rangle$. With $\Delta^2\hat{X}_{\rm in} = \Delta^2\hat{Y}_{\rm in} = 1$ for input coherent state, the fidelity of $\{1,2\}$ reconstruction structure is $F^{\rm clas}_{\{1,2\}} = 1$.

When player1 and player3 are authorized, the secret state can be reconstructed by $\{1,3\}$ reconstruction structure shown in Fig. S2(c). In this reconstruction structure, $\hat{a}_1$ is amplified by a phase-insensitive amplifier (PIA) with the help of $\hat{a}_3$. The output state can be expressed as [44]

$$\hat{a}_{\rm out} = \sqrt{G_2}\hat{a}_1 + \sqrt{G_2-1}\hat{a}^{\dagger}_3. \tag{S7}$$

Under the condition of $G_2 = 2$, $\hat{a}_{\rm out}$ is given by

$$\hat{a}_{\rm out} = \hat{a}_{\rm in} - \hat{a}_{\rm EPR1} + \hat{a}^{\dagger}_{\rm EPR2}. \tag{S8}$$

In this way, the amplitude quadrature and phase quadrature of $\hat{a}_{\rm out}$ are given by

$$\begin{aligned}\hat{X}_{\rm out} &= \hat{X}_{\rm in} - \hat{X}_{\rm EPR1} + \hat{X}_{\rm EPR2},\\ \hat{Y}_{\rm out} &= \hat{Y}_{\rm in} - \hat{Y}_{\rm EPR1} - \hat{Y}_{\rm EPR2}.\end{aligned} \tag{S9}$$

Since the EPR entangled source used here is generated with vacuum injections, $\langle\hat{X}_{\rm out}\rangle = \langle\hat{X}_{\rm in}\rangle$ and $\langle\hat{Y}_{\rm out}\rangle = \langle\hat{Y}_{\rm in}\rangle$. Based on Eqs. (2), (6), and (9), the fidelity of $\{1,3\}$ reconstruction structure can be calculated as [37, 38]

$$F = 2e^{-(k_x+k_y)/4}/\sqrt{(1+\Delta^2\hat{X}_{\rm out})(1+\Delta^2\hat{Y}_{\rm out})} = 1/[2G_1 - 2\sqrt{G_1(G_1-1)}], \tag{S10}$$

where $k_x = k_y = 0$, $\Delta^2\hat{X}_{\rm out} = \Delta^2\hat{Y}_{\rm out} = 4G_1 - 4\sqrt{G_1(G_1-1)} - 1$. Without the help of EPR entangled source ($G_1 = 1$), the maximum achievable fidelity of $\{1,3\}$ reconstruction structure is $F^{\rm clas}_{\{1,3\}} = 1/2$. The $\{2,3\}$ reconstruction structure is equivalent to $\{1,3\}$ structure, and the fidelity of $\{2,3\}$ reconstruction structure ($F^{\rm clas}_{\{2,3\}} = 1/2$) can be calculated similarly. Therefore, the classical fidelity limit of (2, 3) threshold deterministic AOQSS protocol can be given by $F^{\rm clas}_{\rm avg} = \frac{F^{\rm clas}_{\{1,2\}} + F^{\rm clas}_{\{1,3\}} + F^{\rm clas}_{\{2,3\}}}{3} = 2/3$. Based on Eq. (S10), when $G_1 \to \infty$, the average fidelity of (2, 3) threshold deterministic AOQSS protocol $F_{\rm avg} \to 1$, beating the corresponding classical limit.

## 3. The typical results for (2, 3) threshold deterministic AOQSS with $\{2,3\}$ reconstruction structure.

The typical noise power results for (2, 3) threshold deterministic AOQSS with $\{2,3\}$ reconstruction structure are shown in Fig. S3. Similar to AOQSS with $\{1,3\}$ structure, from Fig. S3(a) to Figure S3(d), we can obtain the fidelity ($F_{\{2,3\}} = 0.61 \pm 0.02$) of AOQSS with $\{2,3\}$ structure. The corresponding classical fidelity limit is $0.49 \pm 0.01$, which can be obtained from the orange traces in Fig. S3(a) to Fig. S3(d). Fig. S3(e), (f) shows the noise spectra for the adversary structure $\{1\}$. The obtained fidelity for $\{1\}$ structure is $F_{\{1\}} = 0.09 \pm 0.01$.

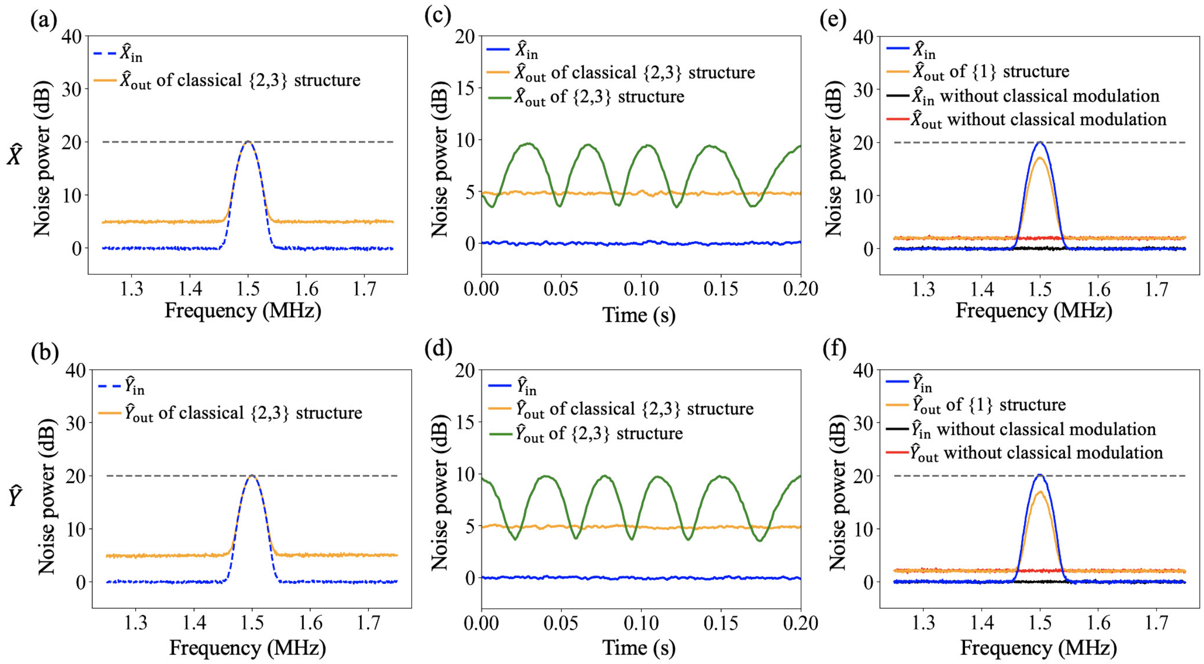


Figure S3: The typical noise power results for $\{2,3\}$ reconstruction structure and the corresponding adversary structure $\{1\}$. (a) and (b) show the amplitude and phase quadrature variances with classical modulations for the input secret state (blue dashed traces) and the output state (orange solid traces) of classical $\{2,3\}$ structure, respectively. (c) and (d) show the amplitude and phase quadrature variances without classical modulations for the input secret state (blue traces), the output state of $\{2,3\}$ structure (green traces), and the corresponding classical $\{2,3\}$ structure (orange traces), respectively. (e) and (f) show the amplitude and phase quadrature variances of $\{1\}$ structure, respectively. The input secret state and the output state with (without) classical modulations are shown as the blue (black) traces and the orange (red) traces, respectively. The vertical scale is normalized to the quadrature variances of the input secret state.

## 4. (3, 5) threshold deterministic AOQSS protocol.

Our scheme provides a platform to implement arbitrary $(k, n)$ threshold AOQSS. Here we show such scalability by giving an example of (3, 5) threshold deterministic AOQSS protocol. Based on Eq. (S1) and Eq. (S2), the two EPR entangled sources used in the dealer protocol shown in Fig. S4 can be expressed as [44]

$$\begin{aligned}
\hat{a}_{\rm E1}(\tau_1) &= \sqrt{G_1}\hat{a}_{01} + \sqrt{G_1 - 1}\hat{b}^{\dagger}_{01}, \\
\hat{b}^{\dagger}_{\rm E1}(\tau_1) &= \sqrt{G_1 - 1}\hat{a}_{01} + \sqrt{G_1}\hat{b}^{\dagger}_{01}, \\
\hat{a}_{\rm E2}(\tau_2) &= \sqrt{G_2}\hat{a}_{02} + \sqrt{G_2 - 1}\hat{b}^{\dagger}_{02}, \\
\hat{b}^{\dagger}_{\rm E2}(\tau_2) &= \sqrt{G_2 - 1}\hat{a}_{02} + \sqrt{G_2}\hat{b}^{\dagger}_{02},
\end{aligned} \tag{S11}$$

where $\hat{a}_{01}$, $\hat{b}_{01}$, $\hat{a}_{02}$, and $\hat{b}_{02}$ are the annihilation operators associated with the vacuum input states. $G_1 = \cosh^2(r_1\tau_1)$ ($G_2 = \cosh^2(r_2\tau_2)$) is the intensity gain of the FWM process and $\tau_1$ ($\tau_2$) is the interaction time. $r_1(r_2)$ denotes the interaction strength of FWM process. $\hat{a}_{\rm E1}$ and $\hat{b}_{\rm E1}$ ($\hat{a}_{\rm E2}$ and $\hat{b}_{\rm E2}$) are the annihilation operators associated with EPR entanglement. Then, based on the dealer protocol of (3, 5) threshold deterministic AOQSS shown in Fig. S4(a), the five shares can be expressed as [38]

$$\begin{aligned}
\hat{a}_1 &= \tfrac{1}{\sqrt{2}}(\hat{a}_{\rm in} + \hat{a}_{\rm E1} + \delta N), \\
\hat{a}_2 &= \tfrac{1}{2}\hat{a}_{\rm in} - \tfrac{1}{2}\hat{a}_{\rm E1} - \tfrac{1}{\sqrt{2}}\hat{a}_{\rm E2} - \tfrac{1}{2}\delta N, \\
\hat{a}_3 &= \tfrac{1}{2}\hat{a}_{\rm in} - \tfrac{1}{2}\hat{a}_{\rm E1} + \tfrac{1}{\sqrt{2}}\hat{a}_{\rm E2} - \tfrac{1}{2}\delta N, \\
\hat{a}_4 &= \tfrac{1}{\sqrt{2}}(\hat{b}_{\rm E2} - \hat{b}_{\rm E1} - \delta N^*), \\
\hat{a}_5 &= \tfrac{1}{\sqrt{2}}(\hat{b}_{\rm E2} + \hat{b}_{\rm E1} + \delta N^*),
\end{aligned} \tag{S12}$$

where $\hat{a}_{\text{in}}$ is the annihilation operator associated with the secret coherent state. $\delta N$ represents the additional Gaussian noise. $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$, $\hat{a}_4$, and $\hat{a}_5$ are the annihilation operators associated with the five shares. Then, five shares are sent to five players.

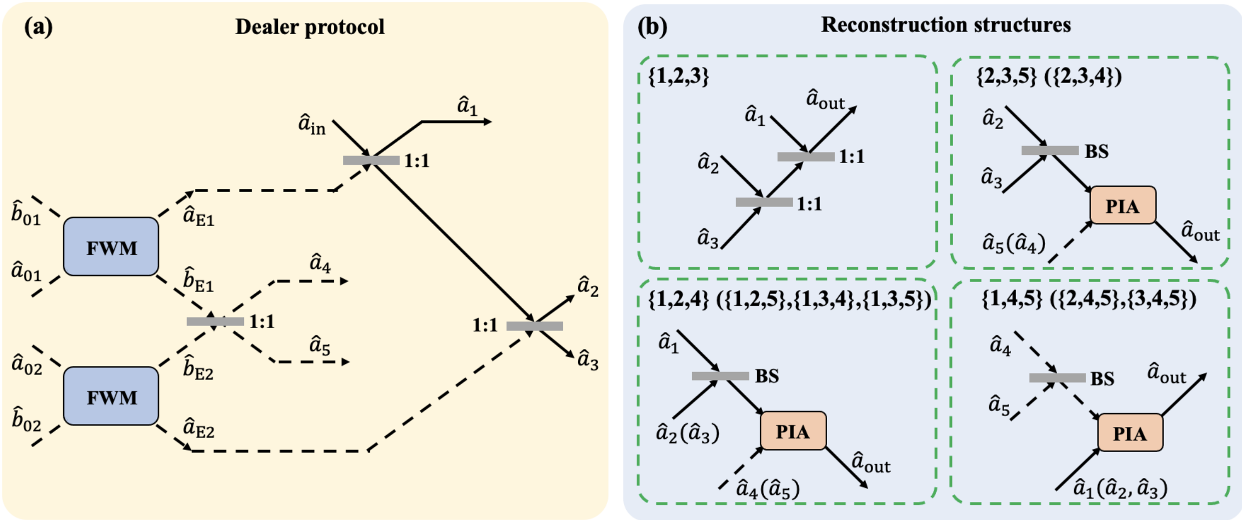


Figure S4: The structure of (3, 5) threshold deterministic AOQSS. (a) The dealer protocol. FWM, four-wave mixing process for generating EPR entanglement; $\hat{a}_{01}$, $\hat{b}_{01}$, $\hat{a}_{02}$, and $\hat{b}_{02}$, the annihilation operators associated with the vacuum input states of the FWM processes; $\hat{a}_{\text{E1}}$ and $\hat{b}_{\text{E1}}$ ($\hat{a}_{\text{E2}}$ and $\hat{b}_{\text{E2}}$), the annihilation operators associated with EPR entanglement; 1:1, 50:50 beam splitter; $\hat{a}_{\text{in}}$, the annihilation operator associated with the secret coherent state; $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$, $\hat{a}_4$, and $\hat{a}_5$, the annihilation operators associated with the five shares. (b) The reconstruction structures. BS, beam splitter. $\hat{a}_{\text{out}}$, the annihilation operator associated with the output reconstructed state; PIA, phase-insensitive amplifier.

For (3, 5) threshold deterministic AOQSS protocol, at least three of these five players need to cooperate to reconstruct the secret state, and there are ten reconstruction structures for (3, 5) threshold deterministic AOQSS as shown in Fig. S4(b). For $\{1, 2, 3\}$ reconstruction structure, $\hat{a}_2$ and $\hat{a}_3$ are combined by a 50:50 BS. The output of this BS $\hat{a}_c$ can be expressed as [37]

$$\hat{a}_c = \frac{1}{\sqrt{2}}(\hat{a}_2 + \hat{a}_3). \tag{S13}$$

To retrieve the secret state, $\hat{a}_c$ and $\hat{a}_1$ are then combined by another 50:50 BS, and the output state $\hat{a}_{\text{out}}$ can be expressed as [37]

$$\hat{a}_{\text{out}} = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_c) = \hat{a}_{\text{in}}. \tag{S14}$$

In this sense, the amplitude quadrature and phase quadrature of the output state are given by

$$\begin{aligned} \hat{X}_{\text{out}} &= \hat{a}^\dagger_{\text{out}} + \hat{a}_{\text{out}} = \hat{X}_{\text{in}}, \\ \hat{Y}_{\text{out}} &= i(\hat{a}^\dagger_{\text{out}} - \hat{a}_{\text{out}}) = \hat{Y}_{\text{in}}. \end{aligned} \tag{S15}$$

Based on Eq. (S6), the fidelity of this $\{1, 2, 3\}$ reconstruction structure can be given as $F^{\text{clas}}_{\{1,2,3\}}$=1.

For $\{1, 4, 5\}$ reconstruction structure, $\hat{a}_4$ and $\hat{a}_5$ are combined by a 50:50 BS. The output of this BS $\hat{a}_c$ can be expressed as [37]

$$\hat{a}_c = \frac{1}{\sqrt{2}}(\hat{a}_4 - \hat{a}_5). \tag{S16}$$

Then, $\hat{a}_1$ is amplified by a PIA with the help of $\hat{a}_c$, which can be expressed as [44]

$$\hat{a}_{\text{out}} = \sqrt{G_3}\hat{a}_1 + \sqrt{G_3 - 1}\hat{a}^\dagger_c, \tag{S17}$$

where $G_3$ is the intensity gain of the PIA. When $G_3=2$, the amplitude quadrature and phase quadrature of the output state and the corresponding quadrature variances can be given as

$$\begin{aligned}\hat{X}_{\text{out}} &= \hat{X}_{\text{in}}+\hat{X}_{\hat{a}_{\text{E1}}}-\hat{X}_{\hat{b}_{\text{E1}}},\\ \hat{Y}_{\text{out}} &= \hat{Y}_{\text{in}}+\hat{Y}_{\hat{a}_{\text{E1}}}+\hat{Y}_{\hat{b}_{\text{E1}}},\\ \Delta^2\hat{X}_{\text{out}} &= \Delta^2\hat{Y}_{\text{out}}=4G_1-4\sqrt{G_1(G_1-1)}-1.\end{aligned} \tag{S18}$$

Based on Eq. (S6), the fidelity of this $\{1,4,5\}$ reconstruction structure can be given as

$$F_{\{1,4,5\}}=1/[2G_1-2\sqrt{G_1(G_1-1)}]. \tag{S19}$$

For $\{1,2,4\}$ reconstruction structure, $\hat{a}_1$ and $\hat{a}_2$ are combined by a BS with a transmissivity of $T$. The output of this BS $\hat{a}_c$ can be expressed as [37]

$$\hat{a}_c=\sqrt{T}\hat{a}_1+\sqrt{1-T}\hat{a}_2. \tag{S20}$$

Then, $\hat{a}_c$ is amplified by a PIA with the help of $\hat{a}_4$, which can be expressed as [44]

$$\begin{aligned}\hat{a}_{\text{out}} &= \sqrt{G_3}\hat{a}_c+\sqrt{G_3-1}\hat{a}_4^{\dagger}\\ &= [\sqrt{\tfrac{G_3T}{2}}+\sqrt{\tfrac{G_3(1-T)}{4}}]\hat{a}_{\text{in}}+[\sqrt{\tfrac{G_3T}{2}}-\sqrt{\tfrac{G_3(1-T)}{4}}]\hat{a}_{\text{E1}}-\sqrt{\tfrac{G_3-1}{2}}\hat{b}_{\text{E1}}^{\dagger}\\ &- \sqrt{\tfrac{G_3(1-T)}{2}}\hat{a}_{\text{E2}}+\sqrt{\tfrac{G_3-1}{2}}\hat{b}_{\text{E2}}^{\dagger}+[\sqrt{\tfrac{G_3T}{2}}-\sqrt{\tfrac{G_3(1-T)}{4}}-\sqrt{\tfrac{G_3-1}{2}}]\delta N.\end{aligned} \tag{S21}$$

When $G_3=7-4\sqrt{2}$ and $T=(4\sqrt{2}+7)/17$, the amplitude quadrature and phase quadrature of the output state can be given as

$$\begin{aligned}\hat{X}_{\text{out}} &= \hat{X}_{\text{in}}+g_{1x}(\hat{X}_{\hat{a}_{\text{E1}}}-\hat{X}_{\hat{b}_{\text{E1}}})+g_{1x}(\hat{X}_{\hat{b}_{\text{E2}}}-\hat{X}_{\hat{a}_{\text{E2}}}),\\ \hat{Y}_{\text{out}} &= \hat{Y}_{\text{in}}+g_{1y}(\hat{Y}_{\hat{a}_{\text{E1}}}+\hat{Y}_{\hat{b}_{\text{E1}}})-g_{1y}(\hat{Y}_{\hat{a}_{\text{E2}}}+\hat{Y}_{\hat{b}_{\text{E2}}}),\end{aligned} \tag{S22}$$

where $g_{1y}=g_{1x}=\sqrt{2}-1$. The corresponding quadrature variances can be expressed as

$$\Delta^2\hat{X}_{\text{out}}=\Delta^2\hat{Y}_{\text{out}}=1+g_{1x}^2[4(G_1+G_2)-4\sqrt{G_1(G_1-1)}-4\sqrt{G_2(G_2-1)}-4]. \tag{S23}$$

Based on Eq. (S6), the fidelity of this $\{1,2,4\}$ reconstruction structure can be given as

$$F_{\{1,2,4\}}=\frac{1}{1+g_{1x}^2[2(G_1+G_2)-2\sqrt{G_1(G_1-1)}-2\sqrt{G_2(G_2-1)}-2]}. \tag{S24}$$

Because $\{1,2,5\}$, $\{1,3,4\}$, and $\{1,3,5\}$ reconstruction structures are equivalent to $\{1,2,4\}$ structure, they have the same fidelity.

By the same token, the fidelities of $\{2,3,5\}$ ($\{2,3,4\}$) and $\{2,4,5\}$ ($\{3,4,5\}$) reconstruction structures can be calculated as

$$\begin{aligned}F_{\{2,3,5\}} &= F_{\{2,3,4\}}=\frac{1}{2(G_1+G_2)-2\sqrt{G_1(G_1-1)}-2\sqrt{G_2(G_2-1)}-1},\\ F_{\{2,4,5\}} &= F_{\{3,4,5\}}=\frac{1}{2G_1-2\sqrt{G_1(G_1-1)}+2[2G_2-2\sqrt{G_2(G_2-1)}-1]}.\end{aligned} \tag{S25}$$

In this way, we can obtain the average fidelity of (3, 5) threshold deterministic AOQSS protocol $F_{\text{avg}}$ by calculating the average fidelity of ten reconstruction structures. Based on Eq. (S19), Eq. (S24), and Eq. (S25), when $G_1\to\infty$ and $G_2\to\infty$, the average fidelity of (3, 5) threshold deterministic AOQSS protocol $F_{\text{avg}}\to 1$, beating the corresponding classical limit of 3/5 ($k/n$ [37, 38]).